Chapter 2

Solving Exponential and Logarithm Problems

2.1 The logarithm is an inverse

In this section we exploit the fact that $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses to get solutions to some problems.

Example 1 *Solve* $e^{2x} = 5$ *for* x*.*

Answer: Since $\ln(e^x) = x$ we take ln of both sides to get.

$$\ln(e^{2x}) = \ln(5).$$

Because $\ln(e^{2x}) = 2x$ this is saying.

$$2x = \ln(e^{2x}) = \ln(5)$$

or just

$$2x = \ln(5).$$

Dividing by 2 we get

$$x = \frac{1}{2}\ln(5).$$

And that is our answer. $\square$

Example 2 *Solve* $7e^{4t} = 33$ *for* t.

Answer: As before we will take ln of both sides. But first, we need to get the e^{Stuff} by itself, so we divide by 7 to get.

$$e^{4t} = 33/7.$$

Because $\ln(e^{4t}) = 4t$ this is saying.

$$4t = \ln(e^{4t}) = \ln(33/7)$$

or just

$$4t = \ln(33/7).$$

Dividing by 4 we get

$$x = \frac{1}{4}\ln(33/7).$$

And that is our answer. □

Now we want to construct an exponential equation that is the analogue of a line,

$$f(t) = Ke^{mt}$$

Example 3 *Find the values of* K *and* m *such that* $f(0) = 2$ *and* $f(3) = 10$.

Answer: First, because $f(0) = 2$, it is easy to find K.

$$2 = f(0) = Ke^{m \cdot 0} = Ke^0 = K \cdot 1 = K.$$

Next, we find m Beginning with $10 = f(3) = 2e^{m \cdot 3}$, we divide by 2 to get

$$\frac{10}{2} = e^{m \cdot 3} \qquad \text{or} \qquad 5 = e^{m \cdot 3}$$

Now solving for m we take the ln of both sides to get

$$\ln(5) = \ln(e^{m \cdot 3} = m \cdot 3$$

Dividing by 3 we get

$$\frac{\ln(5)}{3} = m$$

We have found the values for K and m; we are done. □

This problem was relatively easy because one of the numbers was zero. Let us try something more difficult.

Example 4 *Find the values of K and m such that $f(1) = 2$ and $f(3) = 10$.*

Answer: We begin by writing down the two equations with the indicated values plugged in.

$$2 = f(1) = Ke^{m\cdot 1} = Ke^{m} \quad \text{and}$$

$$10 = f(3) = Ke^{m\cdot 3} = Ke^{3m}.$$

Divide one by the other (we divide the top by the bottom).

$$\frac{10}{2} = \frac{Ke^{3m}}{Ke^{m}}.$$

Now we use the property of exponents, $\dfrac{e^x}{e^y} = e^{x-y}$, to write

$$5 = \frac{10}{2} = \frac{Ke^{3m}}{Ke^{m}} = e^{3m-m}$$

and factor out the m

$$5 = e^{3m-m} = e^{m(3-1)} = e^{2m}.$$

This we know how to solve; take the natural logarithm

$$\ln(5) = \ln(e^{2m}) = 2m.$$

and divide by 2.

$$m = \frac{1}{2}\ln(5).$$

Now that we have m we can go back and use it in something like the first equation,

$$2 = f(1) = Ke^{m\cdot 1} = Ke^{m}$$

to solve for K. Notice what happens if we square both sides of the above equation,

$$2^2 = (Ke^{m})^2 = K^2 \cdot e^{2m}.$$

Also notice that we already solved for $2m$, we got $2m = \ln(5)$. When we put the $\ln(5)$ into the exponent of e we get

$$2^2 = K^2 e^{2m} = K^2 e^{\ln(5)} = K^2 5.$$

Or

$$4 = K^2 \cdot 5.$$

Dividing both sides by 4, we get

$$\frac{4}{5} = K^2$$

or $K = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$.

So, there are our answers:

$$m = \frac{1}{2}\ln(5) \qquad \text{and} \qquad K = \frac{2}{\sqrt{5}}.$$

How do we write the function $f(t) = Ke^{mt}$?

One way to do this is use the $\exp(x)$ function. What is this function? It is just another way to write e^x. For example, $\exp(3) = e^3$ and another example $\exp(\ln(x)) = x$. In short $\exp(x) = e^x$. In other words, we have two different names for the same function. What's the point in that? Well, you may have noticed that m is a little complicated,

$$m = \frac{1}{2}\ln(5).$$

If we try to plug m into e^{mt} we get $e^{\frac{1}{2}\ln(5)t}$. It is a little hard to read. It is much easier to read

$$\exp(\frac{1}{2}\ln(5)t).$$

Using the exp notation our original function looks like

$$f(t) = Ke^{mt}$$

and with our solutions for K and m we get

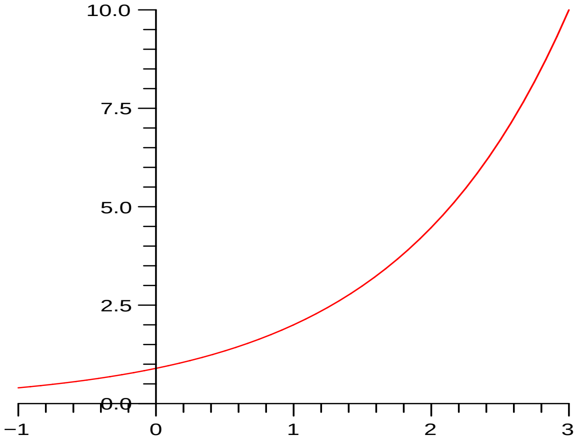
$$f(t) = \frac{2}{\sqrt{5}}\exp(\frac{1}{2}\ln(5)t) = \frac{2}{\sqrt{5}}\exp(t\ln(\sqrt{5})).$$

The graph of $f(x)$ is shown in Figure 2.1

Problem 5

Solve the following for x. $\quad 6 \cdot e^{2x} = 21$

Figure 2.1: The Graph of $f(x) = \frac{2}{\sqrt{5}} \exp(t \ln(\sqrt{5}))$.



Problem 6

Solve the following for x. $\quad 3 \cdot e^{2x} = 21$

Problem 7

Solve the following for x. $\quad 11 \cdot e^{3x} + 2 = 13$

Problem 8

Solve the following for x. $\quad 5 \cdot e^{2x} - 4 = 6$

Problem 9

Solve the following for x. $\quad 5 \cdot e^{0.2x} = 13 \cdot e^{0.3x}$