6.11 Applications of logarithms and exponents

6.11.1 Population models

$P(t) = Ae^{kt}$ with $P(t = 0)$ the initial population. By plugging $t = 0$ we can see that $Ae^{k\cdot 0} = Ae^0 = A$ is the initial population. K is the relative periodic growth rate.

$P(t) = Ae^{kt}$ with the population known at two times $P(t_1) = P_1$ and $P(t_2) = P_2$.

$P(t) = Ab^t$ with $P(t = 0)$ the initial population.

$P(t) = Ab^t$ with the population known at two times $P(t_1) = P_1$ and $P(t_2) = P_2$.

The average growth rate of a population between times t_1 and t_2 is

$$\frac{P(t_2) - P(t_1)}{t_2 - t_1}.$$

The instantaneous population growth rate at time T is $f'(T)$.

6.11.2 Data analysis and Population models

If $P(t) = Ae^{Kt}$ and we have population/time data in the form of pairs of numbers (t_o, P_o) we may want to find values of A and K so that $P(t) = Ae^{Kt}$ is a good representation of the data.

Notice

$$\ln(P(t)) = \ln(Ae^{Kt}) = \ln(A) + \ln(e^{Kt}) = \ln(A) + Kt.$$

If we let $Q = \ln(P)$, then that equation looks like the following.

$$Q(t) = \alpha + Kt$$

where $\alpha = \ln(A)$.

If we take the population data (t_o, P_o) and change the data into $(t_o, \ln(P_o))$ by taking the natural logarithm of the population numbers, then we expect the data to lie on the line $Q = \alpha + Kt$. So, we plot the $(t_o, \ln(P_o))$ data points and find a best fit line. From the best fit line we read off the slope, K, and the intercept α. We solve for A using $\alpha = \ln(A)$. Careful, K is not how much the population ratio will change in one year.

Problems:

1. A population of bacteria is observed to have about 5200 cells in it at 7AM and at 5PM it is observed to have 7200 cells. Construct an exponential model of the bacteria growth. Use this model to predict how many bacteria cells there will be the next day at 7AM.

6.12 Area and the antiderivative of $1/x$ and beyond

We now calculate the values of expressions like the following.

$$\int_1^4 2x^2 + \frac{x}{(x^2+1)}\,dx$$

We know that if we can find the antiderivative $F(x)$ then the answer is just $F(4) - F(1)$. We also know the antiderivative of the x^2 is $x^3/3$ so the answer comes down to figuring out the antiderivative of $\frac{x}{(x^2+1)}$.

Recall we know that if $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$. This tells us that if $g(x) = \frac{1}{x}$, then $G(x) = \ln(x) + C$. So

$$f(x) = \frac{1}{x} \qquad \text{implies} \qquad G(x) = \ln(x) + C.$$

Example 76 Find the antiderivative of $f(x) = \frac{x}{(x^2+1)}$.

Answer: We use substitution. Let $u = x^2 + 1$, then $u' = 2x$, so $\frac{1}{2}u' = x$. Then

$$\widetilde{f}(u) = \frac{\frac{1}{2}u'}{u}.$$

Changing u from a function to a variable makes $u' = 1$, giving us

$$\widetilde{f}(u) = \frac{\frac{1}{2}}{u} = \frac{1}{2}\frac{1}{u}.$$

The antiderivative of this is

$$\widetilde{F}(u) = \frac{1}{2}\ln(u) + C.$$