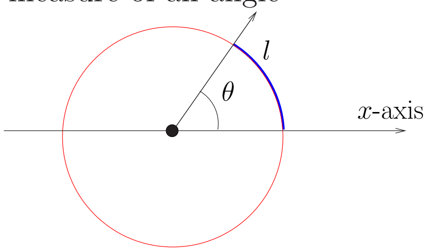
Trigonometry for Calculus with Review

(1) The *Three Pictures of Trigonometry* are associated with the following topics
   (a) Measuring angles
   (b) The definition of sine and cosine
   (c) The graphs of sine and cosine

(2) Picture # 1 is associated with the measurement of angles.

Angles are measured as a fraction of the full circle.

FIGURE 1. The measure of an angle



We have three examples. In the first and most familiar, the convention is that the full circle consists of 360^o (degrees). If the blue arc represents 1/6-th of the full circle, then the measure of the blue arc in degrees is 1/6 th of 360^o, or

$$l = 360^o \cdot \frac{1}{6} = 60^o.$$

The measure of the full circle is up to the civilization that is measuring it. For example, I, leader of my new kingdom, choose to call the unit of angle measurement in my kingdom the *zap*. There are 51 *zaps* in a full circle. From then on, having made that choice, the size of an angle is measured in units of *zaps*. So the above angle is 1/6th of 51 *zaps*, or

$$l = 51\text{zaps} \cdot \frac{1}{6} = 8.5\text{zaps}.$$

This is a rather unnatural unit of angle measurement, but would work perfectly well.

A relatively natural unit of measurement assigns the full circle a size, or measure, of 2π, the circumference of a circle with unit radius. The unit for this measure of an angle is called the *radian* after the radius. In this case, the above blue angle would be 1/6th of 2π, or

$$l = 2\pi\text{radians} \cdot \frac{1}{6} = \frac{\pi}{3}\text{radians}.$$

Conversions

We have just measured l the length of the blue arc in three different units of measurement. If I have found l in one unit, there is an easy way to convert to the value of l in a different unit of measure. The conversion method is based on the fact that angles are measured as fractions of the full circle. So,

$$\frac{\text{arc length}}{\text{full circle}} = \frac{l\,(degrees)}{\text{full circle in }(degrees)} = \frac{l\,(zaps)}{\text{full circle in }(zaps)} = \frac{l\,(radians)}{\text{full circle in }(radians)}.$$

Suppose we have an angle with an arc that measures 67 degrees.
1. How many *zaps* would that angle measure?
2. How many *radians* would that angle measure?

To answer 1. we set up the following equality, where x is the unknown number of *zaps*.

$$\frac{67^o}{360^o} = \frac{x\,(zaps)}{51\,(zaps)}$$

Solving for x we get

$$x\,(zaps) = \frac{67^o}{360^o} \cdot 51\,(zaps)$$

or

$$x\,(zaps) = \frac{51\,(zaps)}{360^o} \cdot 67^o.$$

Staring at this for a bit we realize that it will always work. If we start with an angle measuring $y\,(degrees)$ and we want the convert that measurement to a measurement in the units of *zaps*, we do the following.

$$x\,(zaps) = \frac{51\,(zaps)}{360^o} \cdot y\,(degrees).$$

To answer 2. we do exactly the same process, only the units and the numbers change, *the process remains the same*. We set up the following equality, where x is the unknown number of *radians*.

$$\frac{67^o}{360^o} = \frac{x\,(radians)}{2\pi\,(radians)}$$

Solving for x we get

$$x\,(radians) = \frac{67^o}{360^o} \cdot 2\pi\,(radians)$$

or

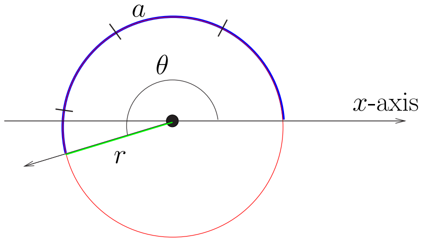
$$x\,(radians) = \frac{2\pi\,(radians)}{360^o} \cdot 67^o.$$

Staring at this for a bit we realize that it will also always work. If we start with an angle measuring $y\,(degrees)$ and we want the convert that measurement to a measurement in the units of *radians*, we do the following.

$$x\,(radians) = \frac{2\pi\,(radians)}{360^o} \cdot y\,(degrees).$$

Some of you have probably memorized this formula before. Me, I can't remember it. I always forget it or get it upside-down. But I can remember the process, and thus don't need to remember the formula. That said, when it comes times to take a quiz or an exam, investing a little time to memorize the formula will probably save a little time on the quiz or exam. But if you forget, well you can get what you need by equating the ratios, as we did here.

FIGURE 2. The meaning of radian measure of an angle



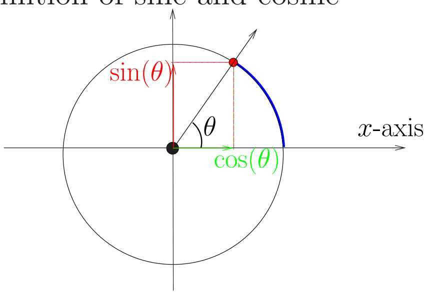
There is another aspect to the radian measure of angle. If you measure the length of the blue arc; call that a, for arc length. And, if you measure the length of the radius; call that r, for radius. Then

$$\theta\,(radians) = \frac{a}{r},$$

no matter what units of length you use to measure the arc length and the radius. In other words the angle θ measured in radians is counting off the length of the arc in multiples of the radius length.

(3) The definition of sine and cosine Picture # 2 is associated with the definition of sine.

FIGURE 3. The definition of sine and cosine



When we want to answer the question, what is the sine of an angle θ, we use Figure 3. To describe what $\sin(\theta)$ is, one finds where the ray corresponding the angle θ intersects the unit circle and puts a big red dot. Then drawing a horizontal red line to the y-axis, the height is $\sin(\theta)$. Notice this is also the height of the red line segment. The $\cos(\theta)$ is found by drawing a green vertical line down to the x-axis. The $\cos(\theta)$ is also the length of the green line segment.

That's it.

Though in general this doesn't really tell us how to find the values of sine and cosine, it just tells us what the numbers mean. If you were *really, really* good with a pencil, you could draw a really careful circle and a really careful 60^o arc, and a really careful red dot and a really careful horizontal line, and really carefully measure the height of the red line, and you would have the $\sin(60^o)$. Similarly for $\cos(60^o)$. Try it.

In class we used a 45^o triangle and an equilateral triangle to work out the values of $sin(\theta)$ for θ equal to 30^o, 60^o, and 90^o. In addition, we used Figure 3 to work out $\sin(\theta$ for θ equal to 0 and 90^o. We then used symmetry to extend this information to those values of θ between 90^o and 180^o. When we were done, we had the following table of data.

(1)

θ^o	0	30	45	60	90	120	135	150	180
$\theta\,(rad)$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

By using symmetry we can extend this to the angles between 180^o and 360^o to get the following table of data.

(2)

θ^o	180	210	225	240	270	300	315	330	360
$\theta\,(rad)$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	π
$\sin(\theta)$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0

Finally, using the same triangles and the same ideas of symmetry we can arrive at a similar table of data for the cosine function.

(3)

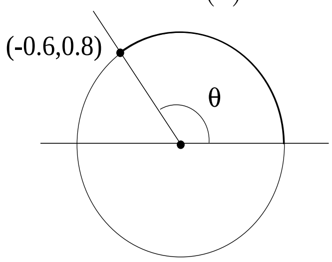
θ^o	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
$\theta\,(rad)$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	π
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

OH MY GOD! No, really it's not that bad. You don't have to memorize it. Just know how to use the triangles to construct the value.

Problems

1. Evaluate exactly:
   a. $\sin(\frac{\pi}{4})$
   b. $\cos(60^o)$
   c. $\cos(120^o)$
   d. $\cos(\frac{3\pi}{4})$
   e. $\sin(0)$
   f. $\cos(\pi)$

2. For the angle θ depicted in the figure to the right, what is the exact value of $\sin(\theta)$?

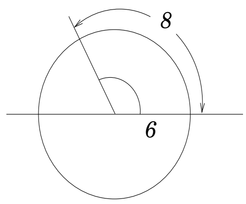


3. Convert 137^o to radians.

4. Convert 247^o to radians (Round to 2 decimal places).

5. A DVD rotates 5/6 of a revolution. How many radians has it travelled?

6. In a circle with radius of 6 yards find the radian measure of the central angle whose terminal side intesects the circle forming an arc which measures 8 yards in length.



7. Sketch the first 2 periods of $f(x) = 1.5\sin(2x)$